Solution Derivations for Capa #9

1) A 48 kg shell is fired from a gun with a muzzle velocity 115 m/s at 55° above the horizontal. At the top of the trajectory, the shell explodes into two fragments of equal mass. One fragment, whose speed immediately after the explosion is zero, falls vertically. What is the horizontal speed of the other fragment?

$m =$ Given
$v =$ Given
$\theta =$ Given
$v_{1x} =$?

Right before the explosion, the velocity of the projectile is only the horizontal component of the initial velocity, or $v_x = v\cos\theta$. Momentum is always conserved, so the initial momentum must equal the final momentum. To simplify the problem (since momentum is a vector quantity), we can consider only the horizontal momentum. Since the particle splits into two equally massed particles, the final momentum is the sum of momenta of the two particles. So,

$$mv_{0x} = \frac{1}{2}mv_{1x} + \frac{1}{2}mv_{2x}.$$

Since one particle completely stops, its horizontal velocity is zero. The equation becomes

$$mv_{0x} = \frac{1}{2}mv_{1x}.$$

The m's cancel, and we're left with

$$v_{0x} = \frac{1}{2}v_{1x}.$$

So,

$$\begin{aligned} v_{1x} &= 2v_{0x} \\ &= 2v_0\cos\theta \end{aligned}$$

2) Two identical steel balls, each of mass 2.6 kg, are suspended from strings of length 34 cm so that they touch when in their equilibrium position. We pull one of the balls back until its string makes an angle $\theta = 54°$ with the vertical and let it go. It collides elastically with the other ball. How high will the other ball rise above its starting point?

$m =$ Given, equal for both balls.
$L =$ Given
$\theta =$ Given

Since the collision is elastic, the second ball will rise to the same height as the first ball. The height from the ball to the mass can be found from a triangle and the height of the ball can be found by subtracting the previous height from the string height.

$$h = L - L\cos\theta = L(1-\cos\theta)$$

This can be shown to be true by conservation of energy:

$$KE_i + PE_i = KE_f + PE_f$$

Since the masses are equal,

$$\begin{aligned} 0 + m_1gh_1 &= 0 + m_2gh_2 \\ mgh_1 &= mgh_2 \\ h_1 &= h_2 \end{aligned}$$

3) Suppose that instead of steel balls we use putty balls . They will collide inelastically and remain stuck together after the collision. How high will the balls rise after the collision?

m = Given
L = Given
θ = Given

Since this collision is inelastic, only conservation of momentum applies. However, we can still use the energy equations but only for the particles before and after (not during) the collision. Momentum is always conserved, so we can always use that equation. In an inelastic collision,

$$m_1v_{1i} + m_2v_{2i} = (m_1+m_2)\,v_f$$

Here, the masses are the same and the second particle has no initial velocity.

$$mv_{1i} = 2mv_f$$

The m's cancel, leaving

$$\frac{1}{2}v_{1i} = v_f \tag{1}$$

In order to get v_{1i}, we can use the energy equation

$$\begin{aligned} \frac{1}{2}mv_{1i}^2 &= mgh \\ v_{1i} &= \sqrt{2gh} \end{aligned}$$

We found an equation for h in the previous problem, so

$$v_{1i} = \sqrt{2gL(1-\cos\theta)}$$

Plugging this into (1), we can solve for v_f

$$\begin{aligned} \frac{1}{2} v_{1i} &= v_f \\ \frac{\sqrt{2gL(1-\cos\theta)}}{2} &= v_f \end{aligned}$$

Now that we know the final velocity of the two masses when they stick together, we can find the height by using the energy equation once again, this time solving for h.

$$\begin{aligned} \frac{1}{2} m v_f^2 &= mgh \\ h &= \frac{v_f^2}{2g} = \frac{\frac{2gL(1-\cos\theta)}{4}}{2g} \end{aligned}$$

Simplifying,

$$h = \frac{2gL(1-\cos\theta)}{8g} = \frac{L(1-\cos\theta)}{4}$$

Remember to convert the units of L to meters or else enter your answer into CAPA as centimeters.

4) A car mass m_1 is moving to the right on a frictionless air track. It collides with a second car, mass m_2, which is initially at rest. Which of the following statements are true? (If A and E are true, and the others are not, enter TFFFT).

QUESTION:
A) If car 1 is much lighter than m_2, and the collision is perfectly elastic, car 1 will continue to the right with nearly its original speed after the collision.
B) If car 1 sticks to car 2, the car1-car2 system must be at rest after the collision.
C) If car 1 sticks to car 2, the final kinetic energy of the car1-car2 system is less than the initial total kinetic energy of the two cars.
D) If the collision is elastic, car 1 must always come to a stop after the collision.
E) Suppose (for this statement only) that car 2 was NOT initially at rest, but was instead heading towards car 1 with equal (but opposite) momentum before the collision. After the collision, the cars must both be at rest.

ANSWER:
a) False. Car 1 will rebound in the opposite direction.
b) False. The system must move to preserve momentum.
c) True. KE not conserved in inelastic collisions.
d) False. It may, but not necessarily.
e) False. Momentum must be conserved, so if the cars had different masses, they would not come to rest.

5) A ball of mass 9.8 g with a speed of 27.5 m/s strikes a wall at an angle 17.0° and then rebounds with the same speed and angle. It is in contact with the wall for 41.0 ms. What is the magnitude of the impulse associated with the collision force?

m = Given
v = Given
θ = Given
t = Given
Impulse = I = ?

$$\begin{aligned} I &= \int_{t_1}^{t_2} F \cdot dt = \Delta p \\ \Delta \mathbf{p} &= mv_f - mv_i \end{aligned}$$

Add the components of the velocities to get the resultant momentum. Assume right is positive x and up is positive y.

$$\begin{aligned} \Delta \mathbf{p} &= mv_{fx} + mv_{fy} - (mv_{ix} + mv_{iy}) \\ &= m(v_{fx} + v_{fy} - v_{ix} - v_{iy}) \end{aligned}$$

Since the initial speed was equal to the final speed, $v = v_f = -v_i$. Since the velocity was broken into components, the sign will determine the direction.

$$\begin{aligned} \Delta \mathbf{p} &= m(-v\sin\theta + v\cos\theta - v\sin\theta - v\cos\theta) \\ &= -2mv\sin\theta \end{aligned}$$

Note that your mass may be given in grams which means you'll need to convert it to kilograms for this problem. Momentum has units $kg \cdot m/s$. CAPA is also looking for the magnitude of the momentum.

6) What is the average force exerted by the ball on the wall?

The average force is (Equation 11-2, pg. 256)

$$\bar{F} = \frac{\Delta p}{\Delta t}$$

Beware! The time may be given in milliseconds (ms), thus you may have to convert it to seconds. Just remember that 1000 ms = 1 s. So,

$$\bar{F} = \frac{-2mv\sin\theta}{\Delta t}.$$

Force is in Newtons (N). Enter the magnitude on this problem, also.

7) A steel ball of mass 0.400 kg is fastened to a cord 61.0 cm long and fixed at the far end. It is released when the cord is horizontal. At the bottom of its path, the ball strikes a 3.80 kg steel block initially at rest on a frictionless surface. The collision is elastic. Find the speed of the ball just after the collision.

$m_1 =$ Given
$L =$ Given
$m_2 =$ Given
$v_{1f} = ?$

In an elastic collision, momentum is conserved but so is kinetic energy. The two equations are

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

and

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

From equations 11-9a and 11-9b,

$$\begin{aligned} v_{1f} &= \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \\ v_{2f} &= \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i} \end{aligned}$$

Since the second particle is initially at rest, and we are solving for the final velocity of the ball, the equations become

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}$$

As before, we can solve for the initial velocity by conservation of energy. Initially, the first mass has only potential energy, but at the bottom of its swing (before it hits the second mass) it has only kinetic energy.

$$\begin{aligned} KE_i + PE_i &= KE_f + PE_f \\ 0 + mgh &= \frac{1}{2} m v_{1i}^2 + 0 \\ 2mgh &= m v_{1i}^2 \\ \sqrt{2gh} &= v_{1i} \end{aligned}$$

The ball is released when the string is vertical, so the height is the length of the string (convert to meters, if necessary). So,

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} \sqrt{2gh}$$

CAPA is looking for the speed, so enter the magnitude of your answer.

8) In a pool game, the cue ball, which has an initial speed of 1.9 m/s, makes an elastic collision with the 8-ball, which is initially at rest. After the collision, the 8-ball moves at an angle $\theta = 17.2°$ with respect to the original direction of the cue ball, as shown in the figure. At what angle ϕ does the cue ball travel after the collision? Assume the pool table is frictionless and the masses of the cue ball and the 8-ball are equal. Use "deg" as units.

$v_{1i} = \text{Given (Cue ball, } m_1)$
$v_{2i} = 0 = \text{Given (8 ball, } m_2)$
$\theta = \text{Given}$
$\phi = ?$

The collision is elastic. Since momentum is a vector, we can split it into horizontal and vertical components. The vertical components must sum to zero and the horizontal components must equal the initial horizontal component. The final kinetic energy must be equal to the initial kinetic energy also. Since the masses are equal, we can write them as a single m. The equations are

$$\begin{aligned} mv_{1ix} + mv_{2ix} &= mv_{1fx} + mv_{2fx} \\ mv_{1iy} + mv_{2iy} &= mv_{1fy} + mv_{2fy} \\ \frac{1}{2}mv_{1i}^2 + \frac{1}{2}mv_{2i}^2 &= \frac{1}{2}mv_{1f}^2 + \frac{1}{2}mv_{2f}^2 \end{aligned}$$

(Since kinetic energy is scalar, there are no components). Notice that in all equations the masses cancel and in the last equation the $\frac{1}{2}$ cancels. Since the second ball (the 8 ball) is initially not moving and the first ball (cue ball) is moving only horizontally, the equations become

$$\begin{aligned} v_{1ix} &= v_{1fx} + v_{2fx} \\ 0 &= v_{1fy} + v_{2fy} \\ v_{1i}^2 &= v_{1f}^2 + v_{2f}^2 \end{aligned}$$

We now have three equations and three unknowns. In order to solve this linear system, we must first relate the final component velocities in terms of the vector sum final velocity. That is, assume the axis (angle $\beta = 0$) runs down the middle of the table. If the angle β is confusing, just remember that the cue ball is only moving in the x (horizontal) direction.

$$\begin{aligned} v_{1ix} &= v_{1i}\cos\beta = v_{1i} \\ v_{1fx} &= v_{1f}\cos\phi \\ v_{1fy} &= v_{1f}\sin\phi \\ v_{2fx} &= v_{2f}\cos\theta \\ v_{2fy} &= v_{2f}\sin\theta \end{aligned}$$

Plugging these new numbers back into the system,

$$\begin{aligned} v_{1i} &= v_{1f}\cos\phi + v_{2f}\cos\theta && (2) \\ 0 &= v_{1f}\sin\phi + v_{2f}\sin\theta && (3) \\ v_{1i}^2 &= v_{1f}^2 + v_{2f}^2 && (4) \end{aligned}$$

The first equation is already solved for v_{1i}, and the second can easily be solved for v_{1f} leaving only v_{2f} in the 3rd equation.

$$v_{1f} = \frac{v_{2f}\sin\theta}{\sin\phi}$$

$$\left(\frac{v_{2f}\sin\theta}{\sin\phi}\cos\phi + v_{2f}\cos\theta\right)^2 = \left(\frac{v_{2f}\sin\theta}{\sin\phi}\right)^2 + v_{2f}^2$$

Expanding this equation,

$$v_{2f}^2\frac{\sin^2\theta}{\sin^2\phi}\cos^2\phi + 2v_{2f}^2\frac{\sin\theta}{\sin\phi}\cos\phi\cos\theta + v_{2f}^2\cos^2\theta = v_{2f}^2\frac{\sin^2\theta}{\sin^2\phi} + v_{2f}^2$$

Solving for v_{2f}^2,

$$\begin{aligned}
v_{2f}^2\frac{\sin^2\theta}{\sin^2\phi}\cos^2\phi + 2v_{2f}^2\frac{\sin\theta}{\sin\phi}\cos\phi\cos\theta + v_{2f}^2\cos^2\theta &= v_{2f}^2\frac{\sin^2\theta}{\sin^2\phi} + v_{2f}^2 \\
v_{2f}^2\left(\frac{\sin^2\theta}{\sin^2\phi}\cos^2\phi + 2\frac{\sin\theta}{\sin\phi}\cos\phi\cos\theta + \cos^2\theta\right) &= v_{2f}^2\left(\frac{\sin^2\theta}{\sin^2\phi} + 1\right) \\
v_{2f}^2\left(\frac{\sin^2\theta}{\sin^2\phi}\cos^2\phi + 2\frac{\sin\theta}{\sin\phi}\cos\phi\cos\theta + \cos^2\theta - \frac{\sin^2\theta}{\sin^2\phi} - 1\right) &= 0
\end{aligned}$$

Since v_{2f}^2 does not equal zero (the 8-ball certainly moves after the collision), the equation becomes

$$\begin{aligned}
\frac{\sin^2\theta}{\sin^2\phi}\cos^2\phi + 2\frac{\sin\theta}{\sin\phi}\cos\phi\cos\theta + \cos^2\theta - \frac{\sin^2\theta}{\sin^2\phi} - 1 &= 0 \\
\frac{\sin^2\theta}{\sin^2\phi}\cos^2\phi + 2\frac{\sin\theta}{\sin\phi}\cos\phi\cos\theta - \frac{\sin^2\theta}{\sin^2\phi} &= 1 - \cos^2\theta \\
\sin^2\theta\cot^2\phi + 2\sin\theta\cos\theta\cot\phi - \frac{\sin^2\theta}{\sin^2\phi} &= \sin^2\theta
\end{aligned}$$

Trig identities $\frac{\cos x}{\sin x} = \cot x$ and $1 - \cos^2\theta = \sin^2\theta$. Divide by $\sin^2\theta$ since $\theta \neq 0$.

$$\begin{aligned}
\cot^2\phi + 2\frac{\cos\theta}{\sin\theta}\cot\phi - \csc^2\phi &= 1 \\
\cot^2\phi + 2\cot\theta\cot\phi - \csc^2\phi &= 1
\end{aligned}$$

Trig identity $\frac{\cos x}{\sin x} = \cot x$

$$\csc^2\phi - 1 + 2\cot\theta\cot\phi - \csc^2\phi = 1$$

Trig identity $\cot^2 x = \csc^2 x - 1$

$$\begin{aligned}
2\cot\theta\cot\phi &= 2 \\
\cot\theta\cot\phi &= 1 \\
\phi &= \cot^{-1}(\tan\theta)
\end{aligned}$$

Remember that $\cot^{-1} x$ can be entered in the calculator as $\arctan\left(\frac{1}{x}\right)$. So,

$$\phi = \arctan\left(\frac{1}{\tan\theta}\right)$$

Or, just remember from lecture (and the vector proof) that $\phi = 90 - \theta$.

9) What is the speed of the cue ball after the collision?

In this problem, we want v_{1f} from the previous problem. We can use the first and last equations of the system above (equations (2) and (4) marked above)

$$\begin{aligned} v_{1i} &= v_{1f}\cos\phi + v_{2f}\cos\theta \\ v_{1i}^2 &= v_{1f}^2 + v_{2f}^2 \end{aligned}$$

Solving the first one for v_{2f},

$$\begin{aligned} v_{1i} - v_{1f}\cos\phi &= v_{2f}\cos\theta \\ \frac{v_{1i} - v_{1f}\cos\phi}{\cos\theta} &= v_{2f} \end{aligned}$$

Plugging v_{2f} into the second equation and solving for v_{1f},

$$\begin{aligned} v_{1i}^2 &= v_{1f}^2 + \left(\frac{v_{1i} - v_{1f}\cos\phi}{\cos\theta}\right)^2 \\ v_{1i}^2 &= v_{1f}^2 + \left(\frac{v_{1i}}{\cos\theta} - \frac{v_{1f}\cos\phi}{\cos\theta}\right)^2 \\ v_{1i}^2 &= v_{1f}^2 + \frac{v_{1i}^2}{\cos^2\theta} - 2\frac{v_{1i}v_{1f}\cos\phi}{\cos^2\theta} + \frac{v_{1f}^2\cos^2\phi}{\cos^2\theta} \\ 0 &= v_{1f}^2\left(1 + \frac{\cos^2\phi}{\cos^2\theta}\right) - v_{1f}\left(2\frac{v_{1i}\cos\phi}{\cos^2\theta}\right) + \frac{v_{1i}^2}{\cos^2\theta} - v_{1i}^2 \\ v_{1f} &= \frac{\left(2\frac{v_{1i}\cos\phi}{\cos^2\theta}\right) \pm \sqrt{\left(2\frac{v_{1i}\cos\phi}{\cos^2\theta}\right)^2 - 4\left(1 + \frac{\cos^2\phi}{\cos^2\theta}\right)\left(\frac{v_{1i}^2}{\cos^2\theta} - v_{1i}^2\right)}}{2\left(1 + \frac{\cos^2\phi}{\cos^2\theta}\right)} \\ v_{1f} &= v_i\frac{\cos\phi + (\cos\theta)\sqrt{(-1 + \cos^2\theta + \cos^2\phi)}}{\cos^2\theta + \cos^2\phi} \end{aligned}$$

(Trust me, it does)

10) Suppose m is a certain mass and v is a certain speed $v > 0$. A mass $m_1 = 5m$ moving to the right with a velocity $v_1 = +9v$ collides head-on with a second mass $m_2 = 5m$ moving to the left with velocity $v_2 = -2v$. The two masses stick together to form a single mass moving with velocity $v_f = x\ v$ where x is a number (no units). What is x?

$m_1 =$ Given
$m_2 =$ Given
$v_{1i} =$ Given
$v_{2i} =$ Given
Since the masses stick, there is not conservation of kinetic energy. However, momentum is conserved.

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2)\, v_f$$

So,

$$\frac{m_1 v_1 + m_2 v_2}{(m_1 + m_2)} = v_f$$

So just plug in the expressions you have for each variable. The m's will end up canceling (leaving only the proportionality constant) and you will end up with a number times v. Your answer is just the number (the coefficient of v).