Solution Derivations for Capa #6

1) Charges are distributed with uniform charge density $\lambda = 4.35\ \mu C/m$ along a semicircle of radius $R = 26.0\ cm$ centered at the origin of a coordinate system (as shown in the diagram below.) What is the potential at the origin?

$\lambda =$ Given
$r =$ Given

The potential difference between two points is defined as

$$\Delta V = -\int_A^B E \cdot dl$$

The first thing to remember is that potential is a scalar. Thus, the direction of E is irrelevant. The length dl away from the charged surface is a constant r. In my case, it was antiparallel to E (the ring is positively charged). This means that the dot product is negative. If E points in the same direction as r, then the dot product will be positive. It will also be easier to switch to polar coordinates when integrating. Thus,

$$\Delta V = -\int_0^\pi E \cdot dl = -\int_0^\pi -rE = \int_0^\pi rE = \int_0^\pi r\frac{k\,dq}{r^2} = \int_0^\pi \frac{k}{r}dq$$

where the last two steps follow because the electric field at each portion of the arc can be regarded as that of a point charge. Since r and k are constants, they can be pulled out of the integral. Thus,

$$\Delta V = \frac{k}{r}\int_0^\pi dq$$

The limits of integration are in terms of θ, but the differential is in terms of charge. We must relate the two.

dq can be found from λ, which is charge per length. Thus, a charge dq equals a length times λ. This length is a portion of the arc over which we are integrating. Thus,

$$\begin{aligned} dq &= \lambda r\,d\theta \\ \Delta V &= \frac{k}{r}\int_0^\pi \lambda r\,d\theta = r\frac{k\lambda}{r}\int_0^\pi d\theta = k\lambda\,(\pi - 0) = \pi\lambda k \end{aligned}$$

2) An isolated metal sphere of radius $10.5\ cm$ is at a potential of $5200\ V$. What is the charge on the sphere?

$r =$ Given
$V =$ Given

The potential for a point charge (which essentially what the sphere reduces to if one is outside the sphere) is

$$V = \frac{kq}{r}$$

Thus,

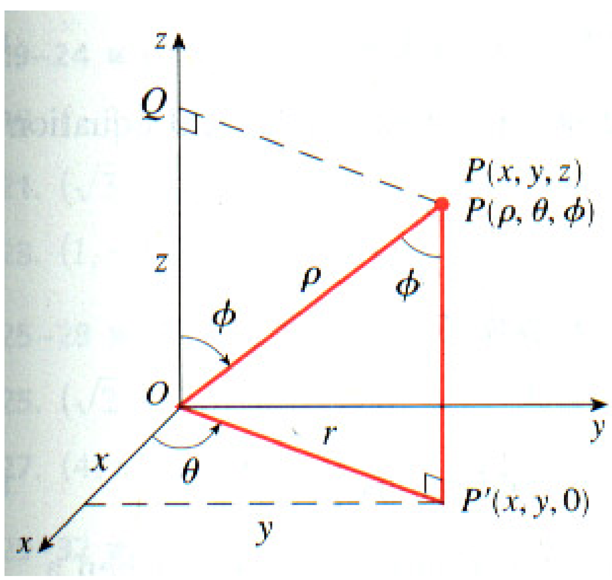
$$q = \frac{rV}{k}$$

3) Determine the energy density of the electric field outside the sphere and integrate this throughout all space in order to calculate the total energy in the electric field.

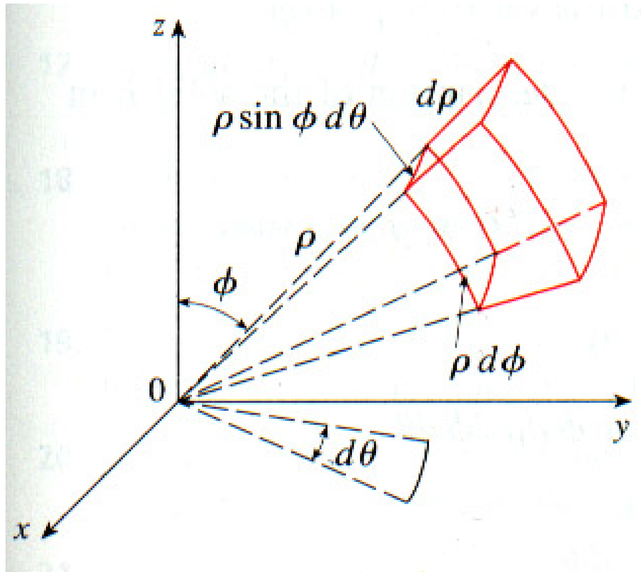
The energy density of an electric field is

$$u_E = \frac{1}{2}\varepsilon_0 E^2$$

which is on page 647 of the textbook. What this question is asking is to find the energy density associated with the electric field in space and add it all up. Thus, you would have to start at all points right outside the sphere's radius, take every point in 3-space, and do a calculation for it. Might as well get started; it will take a while. Actually there is a shortcut way. An integral will do this summation for you. But instead of using the radial vector $\hat{r}$ from the electric field and translating its coordinates into rectangular coordinates, we can transform our variables into spherical coordinates and integrate over all of their range. Since there are three variables in 3-space, we will be performing a triple integral. Without using much Calculus 3, the proof for this integral is a little messy. You can look it up in the calculus 3 portion of your calc book. I am not sure how it is presented, but I have a different calc book and this is the way it was presented there.



This graphic shows a point in 3-space. It is the point at the upper right. As you can see, the rectangular coordinates of the point are $P(x, y, z)$. The spherical representation of the point is also shown. The point is described as a distance from the origin ρ, an angle in the xy-plane θ, and an angle from the z-axis ϕ.



Instead of evaluating the required triple integral over the variables x, y, z, this picture shows how we can do it with the spherical coordinates. The distance from the point charge is ρ. The side of the solid closest to the origin has length $\rho \, d\phi$. This is just an arc length – the radius times the angle. The top of the solid closest to the origin has length $\rho \sin \phi \, d\theta$. To see this, imagine projecting ρ into the xy-plane. The length will be $\rho \sin \phi$. The length of this arc is simply the radius ($\rho \sin \phi$) times the angle ($d\theta$) or just $\rho \sin \phi \, d\theta$. The depth of the box is just the change in the radius from the origin $d\rho$. Instead of integrating over $dx * dy * dz$ we can now use the conversion of $d\rho * \rho \sin \phi d\theta * \rho d\phi$ or just $\rho^2 \sin \phi * d\rho * d\theta * d\phi$. Keep in mind that the radius from the sphere will be represented by ρ.

$$\begin{aligned}
\int\int\int u_E \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi &= \int\int\int \frac{1}{2} \varepsilon_0 E^2 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\
&= \frac{1}{2} \varepsilon_0 \int\int\int E^2 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi
\end{aligned}$$

Before going further, it is important to note the limits of integration. When evaluating multiple integrals, one starts in the center and works outwards. Thus, the limits on the integral furthest inside correspond differential furthest inside. The first differential is $d\rho$ which is the change in radius from the sphere. Its value can be anywhere outside of the sphere itself to infinity. Thus, ρ ranges from r to ∞. To be technically correct, limits of integration cannot range to infinity. The reason is how can you evaluate the definite integral if you don't know what infinity is in order to put it in? The answer is to use a limit. However, I feel the use of one will complicate things more. Just realize that when plugging in infinity after integrating that you are not technically plugging in infinity, but rather evaluating the limit as that variable approaches infinity. The next differential is $d\theta$. This is the angle in the xy-plane. It can be anywhere in the xy-plane as the radius vector can point in any direction. Thus, θ ranges from 0 to 2π. The last differential is $d\phi$. However, ϕ does not range from 0 to 2π like θ. Its range is from 0 to π. To see this, imagine if ϕ was larger than π. This would mean that the vector is now pointing in the other half-space ($y < 0$). However, by letting θ take on values from 0 to 2π this region is already accounted for. Thus, by revolving the vector around the z-axis, ϕ only needs to take on values to get from one side of the z-axis to the other. It is not mandatory that it be set up this way: ϕ could range from 0 to 2π but then θ could only range from 0 to π. The choice is yours. But now, let's put limits on the integrals.

$$\frac{1}{2}\varepsilon_0 \int_0^{\pi}\int_0^{2\pi}\int_r^{\infty} E^2\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

Outside of the sphere, the sphere acts like a point charge. Thus, we can use the formula for the electric field of a point charge, $E = \frac{kq}{r^2}$. Remember, however, that the radius from the sphere in this problem is ρ while the radius of the sphere itself is r. Thus, $E = \frac{kq}{\rho^2}$.

$$\begin{aligned}
& \frac{1}{2}\varepsilon_0 \int_0^{\pi}\int_0^{2\pi}\int_r^{\infty} \left(\frac{kq}{\rho^2}\right)^2 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
= \; & \frac{1}{2}\varepsilon_0 \int_0^{\pi}\int_0^{2\pi}\int_r^{\infty} \frac{k^2q^2}{\rho^4}\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
= \; & \frac{1}{2}\varepsilon_0 \int_0^{\pi}\int_0^{2\pi}\int_r^{\infty} \frac{k^2q^2}{\rho^2} \sin\phi \, d\rho \, d\theta \, d\phi \\
= \; & \frac{1}{2}\varepsilon_0 k^2q^2 \int_0^{\pi}\int_0^{2\pi}\int_r^{\infty} \frac{1}{\rho^2} \sin\phi \, d\rho \, d\theta \, d\phi
\end{aligned}$$

Now comes the part of integrating this which is surprisingly simple once all the constants are removed. Since the integrand is a function of more than one variable but it is inside a multiple integral, we use what is called partial integration. This is similar to partial differentiation in that while integrating with respect to one variable, we hold the rest constant. Thus,

$$\begin{aligned}
= \; & \frac{1}{2}\varepsilon_0 k^2q^2 \int_0^{\pi}\int_0^{2\pi} \left(-\frac{1}{\rho}\sin\phi\right)\Bigg|_{\rho=r}^{\infty} d\theta \, d\phi \\
= \; & \frac{1}{2}\varepsilon_0 k^2q^2 \int_0^{\pi}\int_0^{2\pi} \left(-\frac{1}{\infty}\sin\phi + \frac{1}{r}\sin\phi\right) d\theta \, d\phi \\
= \; & \frac{1}{2}\varepsilon_0 k^2q^2 \int_0^{\pi}\int_0^{2\pi} \frac{1}{r}\sin\phi \, d\theta \, d\phi
\end{aligned}$$

Where the last part follows since the first term approaches zero as the denominator approaches infinity (the technical limit explanation). We can pull the r

outside the integral now since it is a constant for the sphere. Now, we do the second integral.

$$= \frac{1}{2r}\varepsilon_0 k^2 q^2 \int_0^{\pi} \sin\phi\, \theta|_{\theta=0}^{2\pi}\, d\phi$$

This is because when integrating with respect to θ, $\sin\phi$ is treated as a constant. Even though it is a constant, we cannot pull it out of the integral because we still have to integrate it.

$$\begin{aligned}
&= \frac{1}{2r}\varepsilon_0 k^2 q^2 \int_0^{\pi} 2\pi \sin\phi\, d\phi \\
&= \frac{\pi\varepsilon_0}{r} k^2 q^2 \int_0^{\pi} \sin\phi\, d\phi \\
&= \frac{\pi\varepsilon_0}{r} k^2 q^2 \left(-\cos\phi\right)|_{\phi=0}^{\pi} \\
&= \frac{\pi\varepsilon_0}{r} k^2 q^2 \left(-(-1)+1\right) \\
&= 2\frac{\pi\varepsilon_0}{r} k^2 q^2
\end{aligned}$$

I've heard you can also approximate the energy by simply using the equation $U = \frac{1}{2}QV$.

4) Select T-True, F-False, If the first is T and the rest F, enter TFFFF.

QUESTION:
A) The 'drift velocity' of electrons in house-hold electric circuits is close to the speed of light.
B) A negative temperature coefficient of resistivity means that the material is a superconductor.
C) A 5 m length of copper wire has a resistivity of 1.6 $\mu\Omega \cdot cm$. The wire is cut in half. The resistivity of the wire is halved.
D) Current density is a vector.
E) The mean free time between collisions in a current carrying conductor is independent of the applied voltage.

ANSWER:
A) False, the drift velocity is very slow. See lecture notes 27-7.
B) False, a negative temperature coefficient doesn't make sense.
C) False, resistivity is a constant for a given material.
D) True, see page 672-3.
E) True, the equation for drift velocity does not depend on the applied voltage.

Remember to input whether each term is true or false. In this case, FFFTT.

5) At the Aladdin Synchrotron in Madison, Wisconsin, there is an electron beam with a current of 140.0 mA. The electrons have an average kinetic energy of 970 MeV, and a

speed approximately equal to the speed of light. How many electrons pass a given point in the accelerator in one hour? Give your answer in the *numberofelectronsperhour* (i.e. if the answer is 50 then enter "50 1/hr" or "50 hr$\hat{-}$1" into CAPA).

I = Given
KE = Given
v = Given
t = Given
n = ?

For this problem, all you need from the given information is the current. The current is given in milliamperes (that is, 10^{-3}). Current is defined as charge per second. Thus, to find out the amount of electrons per second flowing through, we need to divide by the charge of an electron.

$$\frac{electrons}{second} = I * \frac{1\ electron}{1.6 \times 10^{-19}\ C}$$

Now, to find the amount going through in one hour, simply multiply by 1 hour (3600 seconds).

6) Assume that copper has one electron per atom to carry charge. Given that the mass density of copper is $8.9 \times 10^3\ kg/m^3$ and that its molecular weight is 63.5 g/mol, calculate the drift speed of the electrons in a copper wire that carries 1.20 A and has a circular cross section 0.80 mm in radius.

d = density = Given
w = weight = Given
I = Given
r = Given
v_d = ?

From the book (p.671) we get the equation

$$I = nqAv_d$$

Solving for v_d,

$$v_d = \frac{I}{nqA}$$

We must now calculate the number of electrons present. Using the density and the molecular weight, we can find the number of atoms per volume. To see this, consider density / molecular weight $= \frac{kg}{m^3}/\frac{kg}{mol} = \frac{kg}{m^3} * \frac{mol}{kg} = \frac{mol}{m^3}$. Thus, $n = \frac{d}{w}$. However, this tells us the number of moles of atoms per unit volume. We want the number of atoms (and ultimately the number of electrons). We must multiply by Avogadro's Number which is 6.02×10^{23} atoms/mol. So,

$$n = \frac{d}{w}\left(6.02 \times 10^{23}\right)$$

This gives us number of atoms per unit volume. Since each atom contributes an average of one electron, this is also the number of electrons per unit volume. The charge of an electron, q, is $1.6 \times 10^{-19}\ C$. The cross sectional area is just πr^2, but remember to convert the radius to meters. Since we know I and q and have calculated n and A, we can now find v_d.

7) A number 16 copper wire has a diameter of 1.291 mm. Calculate the resistance of a 31 m long piece of that wire. Use $\rho = 1.72 \times 10^{-8}\ \Omega \cdot m$ for the resistivity of copper. Use "Ohm" as your units, i.e. if the answer is 5 ohms, enter "5 Ohm."

$d =$ Given
$l =$ Given
$\rho =$ Given

Resistance is given by the equation (from lecture notes 27-4)

$$R = \frac{\rho l}{A}$$

A is easily calculated and the others are given. Note that you are given a diameter in mm.

8) For safety, the National Electrical Code limits the allowable amount of current which such a wire may carry. When used in indoor wiring, the limit is 6 A for rubber insulated wire of that size. How much power would be dissipated in the wire of the above problem when carrying the maximum allowable current?

$I =$ Given

Power has many forms. The one of interest here is

$$P = I^2 R$$

Using the value from the previous problem, simply plug in the numbers. Power has units of watts.

9) What would be the voltage between the ends of the wire in the last problem?

The equation needed is

$$V = IR$$

10) The graph represents the Voltage-Current characteristic for an unknown resistor. Determine the value of the resistor from the graph. Use "Ohm" as your units, i.e. if the answer is 5 ohms, enter "5 Ohm."

Get out the vernier calibers... You need to find the slope of that line. Note that the vertical axis is given times 10^3.

11) A typical household circuit is capable of carrying 15.0 A of current at 120 V before the circuit breaker will trip. How many 1500 W hair dryers can run off one such circuit? (Give your answer as a unitless integer.)

I = Given
V = Given
P_{hair} = Given (amount of power each hair dryer uses)

For this problem, it will be convenient to use the following equation for power

$$P = IV$$

This is the total power that the circuit can handle. Divide this number by the power required by each hair dryer to find the amount of hair dryers you can run.

12) In the rush to get ready for lecture, a physics professor leaves the hairdryer described in the previous problem running and does not turn it off until he gets home 6.4 hr later. How much will this add, in dollars, to his next electric bill (assume electricity costs \$0.078 per $kWhr$)? Enter the number of dollars without units; i.e. if the answer is 50 cents, enter "0.50."

t = Given (hours)
c = Given (price of electricity)
P_{hair} = Given (previous problem)

In order to find the cost of running the hair dryer, we must multiple the amount of power used by the hair dryer. (dollar/kWhr * kWhr = dollar) In order to find the power used by the hair dryer, we can calculate the amount of kilowatt hours used. Thus the power used is

$$P = P_{hair} * t$$

and the cost is cost = cP

13) A 12 V lead-acid car battery, engineered for 'up to 500 or more charge/discharge cycles,' has a rating of 95.0 $A \cdot hr$. (It sells for \$126.00.) Calculate the total amount of charge that moves through the battery before it needs to be recharged.

V = Given
rating = Given

The rating means that the battery can deliver the specified amount of amps for one hour (or half as many for twice as long, etc.). Since current is charge/time, the amount of charge is the current times the time. But that is basically given to you in the rating. However, since current is defined as charge per second, you must convert the hour to seconds and then multiply.
charge = rating * time

14) What is the total electrical energy that the battery can deliver before it needs to be recharged?

Power is defined to be energy divided by time. So, energy would then be power times time. Power can be calculated using the previously given information.

$$\begin{aligned} P &= IV \\ E &= P * time \end{aligned}$$

The current and time are related. You know the rating, so you can come up with any combination of current and time that will yield that rating. Thus, the full amperage for one hour, half for twice as long, etc.

15) An underground telephone cable, consisting of a pair of wires, has suffered a short somewhere along its length (at point P in the Figure). The telephone cable is 5.00 km long, and in order to determine where the short is, a technician first measures the resistance between terminals AB; then he measures the resistance across the terminals CD. The first measurement yields 25.00 Ω; the second 100.0 Ω. Where is the short? Give your answer as a distance from point C.

This problem is very general and difficult to explain.
d_{tot} = Given
R_1 = Given (resistance across AB)
R_2 = Given (resistance across CD)
We know that resistance is defined as

$$R = \frac{\rho l}{A}$$

Let d_1 be the distance to the short from the terminals AB. Let d_2 be the distance to the short from the terminals CD. Since the total wire length is given, we know that

$$d_1 + d_2 = d_{tot} \tag{1}$$

Next, set up the equation for the resistance at each point

$$R_1 = \frac{\rho d_1}{A} \tag{2}$$

$$R_2 = \frac{\rho d_2}{A} \tag{3}$$

Since d_1 and d_2 are the unknowns (technically ρ and A are constants), we can solve for one of them and plug it into the other.

$$d_2 = \frac{R_2 A}{\rho} \tag{4}$$

From (1),

$$d_1 = d_{tot} - d_2$$

Thus,

$$d_1 = d_{tot} - \frac{R_2 A}{\rho}$$

Plugging back into (2),

$$R_1 = \frac{\rho\left(d_{tot} - \frac{R_2 A}{\rho}\right)}{A}$$

$$AR_1 = \rho d_{tot} - R_2 A$$

$$AR_1 + R_2 A = \rho d_{tot}$$

$$A\left(R_1 + R_2\right) = \rho d_{tot}$$

$$A = \frac{\rho d_{tot}}{(R_1 + R_2)}$$

We can now plug this back into equation (4) and solve for the required distance from terminal CD.

$$d_2 = \frac{R_2 \frac{\rho d_{tot}}{(R_1+R_2)}}{\rho}$$

$$d_2 = R_2 \frac{d_{tot}}{(R_1 + R_2)}$$

Remember to convert the total distance to meters.